Splitting-up Technique and Cubic Spline Approximations For Solving Modified Coupled Burgers' Equations

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Abstract

In this paper, a finite difference scheme based on the splitting-up technique and cubic spline approximations is developed for solving modified coupled Burgers' equations. The accuracy and stability of the scheme have been analyzed. It is found that the scheme is of first-order accuracy in time and second-order accuracy in space direction and is unconditionally stable. The numerical results are obtained with severe/moderate gradients in the initial and boundary conditions and the steady state solutions are plotted for different values of given parameters. It is concluded that the resulting scheme produces satisfactory results, even in the case of very severe gradient in the solution, and is applicable at both low and high Reynolds numbers. Also, the general nature of the proposed scheme provides a wider application in the solution of non-linear problems arising in mechanics and other areas.

Keywords: Finite Difference Scheme, Operator Splitting, Cubic Spline, Modified Coupled Burgars' equations.

Introduction:

In the last few years, the study of Burgers' equation has been the object of considerable attention. This equation exhibits great similarity with the Navier-Stokes equations, so it is often arises in the mathematical modeling used to solve problems in fluid dynamics involving turbulence [4], and also arises in the approximate theory of flow through a shock wave propagation in a viscous fluid [6]. In one and two space dimensions, Burgers' equations are, respectively,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\lambda\frac{\partial^2 u}{\partial x^2}=0, \qquad (1)$$

and

$$\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0;\\&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\lambda\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0,\end{aligned} \qquad (2)$$

where λ is the kinematic viscosity.

Two-dimensional Burgers' equations are an appropriate test case because the equations structure are similar to that of the incompressible fluid flow momentum equations. This system of equations is used in models for the study of hydrodynamical turbulence and wave processes in a non-linear thermoelastic medium. These equations have also been used as model equation for comparing the accuracy of different computational algorithms developed for non-linear problems by several authors (see, for example, Arminjon and Beauchamp [1], Iyenger and Pillai [8], Jain, Shankar & Singh [9], Rubin and Graves [11], Bassaif [5], Soliman [13], Manoj and Sapna [10] , Sarboland and Aminataei [14] etc., and the references cited therein).

Difficulties have been experienced in the past in the numerical solution of Burgers' equation for small values of the parameter v. During recent years, many authors have used a variety of numerical techniques in attempting to solve the equation for small values of v, which correspond to steep fronts in the propagation of dynamic wave forms. Rubin and Graves [11] have used spline function technique and quasilinearization for the numerical solution of Burgers' equation in one space dimension at low Reynolds numbers. For two-dimensional Burgers' equations, they have proposed a Spline-Alternating-Direction-Implicit (SADI) method for solving the problem at low Reynolds numbers. An extension of this method for solving coupled Burgers' equations gives rise to a complicated numerical scheme which is less efficient compared to the method proposed in the present paper, which is applicable at both low and high Reynolds numbers.

It is worth mentioning here that, the study of non-linear higher-order partial differential equations in two and three space dimensions will provide the most interesting and rewarding problems for computational mathematics of the future. Moreover, challenging non-linear problems involve high discontinuity and therefore one should choose an appropriate model to take care of non-linearity with initial and boundary conditions having internal or

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boundary gradients. This will make them more representative of real fluid dynamic problems. Keeping this in view, in the present paper, we have considered the numerical solution of "Modified Coupled Burgers' Equations" in which a non-linear source term is also included. These equations are presented in the next Section. The detailed plan of this paper is as follows:
In Section 2, the modified coupled Burgers' equations are given. Using a five-step splitting-up technique and cubic spline approximations, the finite difference scheme is derived. In Section 3, the stability and accuracy analysis of the sheme are analyzed. It is found that the scheme is unconditionally stable and of first-order accuracy with respect to time and second-order accuracy with respect to space direction. In Section 4, the numerical results and discussion are discussed. Finally, the conclusion of this study is reported in Section 5. It is concluded that, the scheme is computationally efficient and produces satisfactory results in the ease of very severe gradients in the solutions, and it is applicable at both low and high Reynolds numbers.

Differential Equations and Numerical Scheme :

In this Section, the modified coupled Burgers' equations, under consideration, are given as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=u-2\mathrm{Re}\,u\sqrt{u^2+v^2}\ ;$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=v-2\mathrm{Re}\,v\sqrt{u^2+v^2}\ , \qquad (3)$$

subject to the initial conditions:

$u(x,y,0)=f_1(x,y)$; $(x,y)\in D$;

$v(x,y,0)=f_2(x,y)$; $(x,y)\in D$,

and the boundary conditions:

$u(x,y,t)=g_1(x,y,t)$; $(x,y)\in\partial D$; $t>0$

$v(x,y,t)=g_2(x,y,t)$; $(x,y)\in\partial D$; $t>0$,

where D = $\{(x,y)) : 0\le x,y\le 1\}$, ∂D is its boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined and f_1, f_2, g_1, g_2 are known functions. Re is the Reynolds number often arises in the mathematical modeling used to solve problems in fluid dynamics involving turbulence, the reciprocal of which is considered to be the kinematic viscosity. Furthermore, the discrete approximation for the velocity components u and v at the mesh point $(x_i=ih, y_j=jh, t=nk)$ are denoted by $U^n_{i,j}$ and $V^n_{i,j}$, respectively $(i,j=0,1,2,...,N; n=0,1,2,...)$, where h is the mesh step in direction x and y, and k is the increment in time.

By using the splitting-up technique and cubic spline approximations, the aforementioned modified coupled Burgers' equations, given by (3), can be solved numerically as follows:
We split the first equation of (3), by using a five-time-step splitting technique, in the form:

$$\frac{1}{5}\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}, \qquad (4)$$

$$\frac{1}{5}\frac{\partial u}{\partial t}=-v\frac{\partial u}{\partial y}, \qquad (5)$$

$$\frac{1}{5}\frac{\partial u}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad (6)$$

$$\frac{1}{5}\frac{\partial u}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}, \qquad (7)$$

$$\frac{1}{5}\frac{\partial u}{\partial t}=u-2\mathrm{Re}\,u\sqrt{u^2+v^2}\ . \qquad (8)$$

In equations (4) and (5) the space derivatives are approximated by the first-order derivative of a cuble spline function $s_n(x)$ interpolating $U^n_{i,j}$ $(i,j=0,1,2,...,N)$ at n and $(n+1/5)$ time level, and the time derivatives are approximated by the forward differences. Hence, equation (4) becomes

$$\theta_1 m_{i,j}^{n+1/5} + (1-\theta_1) m_{i,j}^{n} = -\frac{(U_{i,j}^{n})^{-1}}{k}(U_{i,j}^{n+1/5} - U_{i,j}^{n}), \tag{9}$$

where $0 \le \theta_1 \le 1$ and $m_{i,j}^{n}$ denotes the first-order derivative of the cubic spline function $s_n(x)$.

Now, from the condition of continuity of the cubic spline function $s_n(x)$,we have the following spline relation [2]:

$$m_{i+1,j}^{n} + 4m_{i,j}^{n} + m_{i-1,j}^{n} = \frac{1}{2h}\delta_x U_{i,j}^{n}, \tag{10}$$

where $\delta_x U_{i,j}^{n} = U_{i+1,j}^{n} - U_{i-1,j}^{n}$.

By making use of (10),we eliminate the space derivatives from equation (9), and after some mathematical simplifications, we obtain the finite difference approximation to equation (4) in the form:

$$\left\{1 + \frac{1}{6}U_{i,j}^{n}\delta_x^2 (U_{i,j}^{n})^{-1} + \frac{rh}{2}\theta_1 U_{i,j}^{n}\delta_x\right\} \left(U_{i,j}^{n+1/5} - U_{i,j}^{n}\right) = -\frac{rh}{2}U_{i,j}^{n}\delta_x U_{i,j}^{n}, \tag{11}$$

where $r = k/h^2$ and $\delta_x^2 U_{i,j}^{n} = U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i+1,j}^{n}$.

Similarly, the finite difference approximation to equation (5), at $n+2/5$ and $n+1/5$ time levels, will be.

$$\left\{1 + \frac{1}{6}V_{i,j}^{n}\delta_x^2 (V_{i,j}^{n})^{-1} + \frac{rh}{2}\theta_2 V_{i,j}^{n}\delta_y\right\} \left(U_{i,j}^{n+2/5} - U_{i,j}^{n+1/5}\right) = -\frac{rh}{2}V_{i,j}^{n}\delta_y U_{i,j}^{n}, \tag{12}$$

where $0 \le \theta_2 \le 1$.

Further, in equations (6) and (7), the space derivatives are approximated by the second-order derivative of the cubic spline function $S_n(x)$ interpolating $U_{i,j}^{n}$ $(i,j = 0,1,2,...,N)$ at $n+3/5$ and $n+2/5$ time levels, and the time derivatives are approximated by the forward differences. Hence, equation (6) takes the form:

$$\frac{1}{k}(U_{i,j}^{n+3/5} - U_{i,j}^{n+2/5}) = \frac{1}{\mathrm{Re}}\left\{\theta_3 M_{i,j}^{n+3/5} + (1-\theta_3) M_{i,j}^{n+2/5}\right\}, \tag{13}$$

where $0 \le \theta_3 \le 1$ and $M_{i,j}^{n}$ denotes the second-order derivative of the cubic spline function $S_n(x)$. From the condition of continuity, we are taking the following spline relation [2]:

$$M_{i+1,j}^{n} + 4M_{i,j}^{n} + M_{i-1,j}^{n} = \frac{1}{h^2}\delta_x^2 U_{i,j}^{n}. \tag{14}$$

Using (14), we eliminate the space derivatives from equation (13), and after simplifying, we obtain the following finite difference approximation to equation (6):

$$\left\{1 + \left(\frac{1}{6} - \frac{r\theta_3}{\mathrm{Re}}\right)\delta_x^2\right\}(U_{i,j}^{n+3/5} - U_{i,j}^{n+2/5}) = \frac{r}{\mathrm{Re}}\delta_x^2 U_{i,j}^{n+2/5}. \tag{15}$$

In the same manner we can obtain the finite difference approximation to equation (7), at $n+4/5$ and $n+3/5$ time levels, as follows:

$$\left\{1 + \left(\frac{1}{6} - \frac{r\theta_4}{\mathrm{Re}}\right)\delta_y^2\right\}(U_{i,j}^{n+4/5} - U_{i,j}^{n+3/5}) = \frac{r}{\mathrm{Re}}\delta_y^2 U_{i,j}^{n+3/5}, \tag{16}$$

where $0 \le \theta_4 \le 1$. For equation (8), we can write the finite difference approximation, at $n+1$ and $n+4/5$ time levels, in the form.

$$\frac{1}{k}(U_{i,j}^{n+1} - U_{i,j}^{n+4/5}) = U_{i,j}^{n+4/5} - 2\,\mathrm{Re}\,U_{i,j}^{n+4/5}\left\{(U_{i,j}^{n+4/5})^2 + (V_{i,j}^{n+4/5})^2\right\}^{1/2}. \tag{17}$$

Following the above same procedures, we can obtain the corresponding finite difference approximations, for the second equation of (3), in the following forms:

$$\left\{1+\frac{1}{6}V_{i,j}^{n}\delta_y^2(V_{i,j}^{n})^{-1}+\frac{rh}{2}\theta_1 V_{i,j}^{n}\delta_y\right\}(V_{i,j}^{n+1/5}-V_{i,j}^{n})=-\frac{rh}{2}V_{i,j}^{n}\delta_y V_{i,j}^{n}, \quad (18)$$

$$\left\{1+\frac{1}{6}U_{i,j}^{n}\delta_x^2(U_{i,j}^{n})^{-1}+\frac{rh}{2}\theta_2 U_{i,j}^{n}\delta_x\right\}(V_{i,j}^{n+2/5}-V_{i,j}^{n+1/5})=-\frac{rh}{2}U_{i,j}^{n}\delta_x V_{i,j}^{n+1/5}, \quad (19)$$

$$\left\{1+\left(\frac{1}{6}-\frac{r\theta_3}{\mathrm{Re}}\right)\delta_y^2\right\}(V_{i,j}^{n+3/5}-V_{i,j}^{n+2/5})=\frac{r}{\mathrm{Re}}\delta_y^2 U_{i,j}^{n+2/5}, \quad (20)$$

$$\left\{1+\left(\frac{1}{6}-\frac{r\theta_4}{\mathrm{Re}}\right)\delta_x^2\right\}(V_{i,j}^{n+4/5}-V_{i,j}^{n+3/5})=\frac{r}{\mathrm{Re}}\delta_x^2 U_{i,j}^{n+3/5} \quad (21)$$

$$\frac{1}{k}(V_{i,j}^{n+1}-V_{i,j}^{n+4/5})=V_{i,j}^{n+4/5}-2\,\mathrm{Re}\,V_{i,j}^{n+4/5}\left\{(U_{i,j}^{n+4/5})^2+(V_{i,j}^{n+4/5})^2\right\}^{1/2}. \quad (22)$$

Thus, the above ten equations (11), (12), (15), (16), (17) and (18) to (22) are the multi-step finite difference formulation of the modified coupled Burgers' equations, given by (3) .
Now, the intermediate values included in equations (11), (12), (15) and (16) have been taken as:

$$U_{i,j}^{n+1/5}=\left(1-\frac{rh}{2}U_{i,j}^{n}\delta_x\right)U_{i,j}^{n} \qquad i=0,N;\quad j=0,1,2,...,N,$$

$$U_{i,j}^{n+2/5}=\left(1-\frac{rh}{2}V_{i,j}^{n}\delta_y\right)U_{i,j}^{n+1/5} \qquad j=0,N;\quad i=0,1,2,...,N,$$

$$U_{i,j}^{n+3/5}=\left(1-\frac{r}{\mathrm{Re}}\delta_x^2\right)U_{i,j}^{n+2/5} \qquad i=0,N;\quad j=0,1,2,...,N,$$

$$U_{i,j}^{n+4/5}=\left(1-\frac{r}{\mathrm{Re}}\delta_y^2\right)U_{i,j}^{n+3/5} \qquad j=0,N;\quad i=0,1,2,...,N,$$

and the intermediate values included in equation (18)- (21) have been taken as

$$V_{i,j}^{n+1/5}=\left(1-\frac{rh}{2}V_{i,j}^{n}\delta_y^2\right)V_{i,j}^{n} \qquad j=0,N;\quad i=0,1,2,...,N,$$

$$V_{i,j}^{n+2/5}=\left(1-\frac{rh}{2}U_{i,j}^{n}\delta_x\right)V_{i,j}^{n+1/5} \qquad i=0,N;\quad j=0,1,2,...,N,$$

$$V_{i,j}^{n+3/5}=\left(1+\frac{r}{\mathrm{Re}}\delta_y^2\right)V_{i,j}^{n+2/5} \qquad j=0,N;\quad i=0,1,2,...,N,$$

$$V_{i,j}^{n+4/5}=\left(1+\frac{r}{\mathrm{Re}}\delta_x^2\right)V_{i,j}^{n+3/5} \qquad i=0,N;\quad j=0,1,2,...,N.$$

We replace δ_x and δ_x^2 at the lower boundary $i=0$ $(2\Delta_x-\Delta_x^2)$ and Δ_x^2, respectively, and at the upper boundary $i=N$, δ_x and δ_x^2 are replaced by $(2\nabla_x-\nabla_x^2)$ and ∇_x^2 , respectively. Similarly, we can write down corresponding expressions for δ_y and δ_y^2 . Here, the forward and backward operators are defined, respectively, as follows:

$$\Delta_x U_{i,j}^{n}=U_{i+1,j}^{n}-U_{i,j}^{n},$$
$$\nabla_x U_{i,j}^{n}=U_{i,j}^{n}-U_{i-1,j}^{n}.$$

Stability and Accuracy Analysis of the Scheme:
In order to analyze the stability and accuracy of the scheme, we eliminate the intermediate values, and by performing the necessary simplifications, the scheme finally takes the form:

$$W^{n+1} = QW^n + H , \tag{23}$$

where

$$W^n = \begin{bmatrix} U^n_{i,j} \\ V^n_{i,j} \end{bmatrix}, \qquad Q = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

in which C_1 and C_2 are given by:

$$C_1 = \left[1 + k(1 - 2\,\mathrm{Re})\right] \left[A^2 + B^2\right]^{1/2} A,$$

$$C_2 = \left[1 + k(1 - 2\,\mathrm{Re})\right] \left[A^2 + B^2\right]^{1/2} B,$$

where

$$A = \left(1 - \frac{rh}{2} U^n_{i,j}\delta_x\right) \left(1 - \frac{rh}{2} V^n_{i,j}\delta_y\right) \left(1 + \frac{r}{\mathrm{Re}}\delta_x^2\right) \left(1 + \frac{r}{\mathrm{Re}}\delta_y^2\right) U^n_{i,j},$$

$$B = \left(1 - \frac{rh}{2} V^n_{i,j}\delta_y\right) \left(1 - \frac{rh}{2} U^n_{i,j}\delta_x\right) \left(1 + \frac{r}{\mathrm{Re}}\delta_y^2\right) \left(1 + \frac{r}{\mathrm{Re}}\delta_x^2\right) V^n_{i,j}.$$

In the amplification matrix Q, D_1 and D_2 are given by

$$D_1 = \frac{\begin{gathered}\left[1 + \left(\frac{1}{6} - \frac{r}{\mathrm{Re}}(\theta_2 - 1)\right)\delta_x^2\right] \left\{1 + U^n_{i,j}\left[\frac{1}{6}\delta_x^2 (U^n_{i,j})^{-1} + \frac{rh}{2}(\theta_1 - 1)\delta_x\right]\right\} \\ \times \left[1 + \left(\frac{1}{6} - \frac{r}{\mathrm{Re}}(\theta_4 - 1)\right)\delta_y^2\right] \left\{1 + V^n_{i,j}\left[\frac{1}{6}\delta_y^2 (V^n_{i,j})^{-1} + \frac{rh}{2}(\theta_3 - 1)\delta_y\right]\right\}\end{gathered}}{\begin{gathered}\left[1 + \left(\frac{1}{6} - \frac{r\theta_2}{\mathrm{Re}}\right)\delta_x^2\right] \left\{1 + U^n_{i,j}\left[\frac{1}{6}\delta_x^2 (U^n_{i,j})^{-1} + \frac{rh}{2}\theta_1\delta_x\right]\right\} \\ \times \left[1 + \left(\frac{1}{6} - \frac{r\theta_4}{\mathrm{Re}}\right)\delta_x^2\right] \left\{1 + V^n_{i,j}\left[\frac{1}{6}\delta_y^2 (V^n_{i,j})^{-1} + \frac{rh}{2}\theta_3\delta_y\right]\right\}\end{gathered}},$$

$$D_2 = \frac{\begin{gathered}\left[1 + \left(\frac{1}{6} - \frac{r}{\mathrm{Re}}(\theta_2 - 1)\right)\delta_y^2\right] \left\{1 + V^n_{i,j}\left[\frac{1}{6}\delta_y^2 (V^n_{i,j})^{-1} + \frac{rh}{2}(\theta_1 - 1)\delta_y\right]\right\} \\ \times \left[1 + \left(\frac{1}{6} - \frac{r}{\mathrm{Re}}(\theta_4 - 1)\right)\delta_x^2\right] \left\{1 + U^n_{i,j}\left[\frac{1}{6}\delta_x^2 (U^n_{i,j})^{-1} + \frac{rh}{2}(\theta_3 - 1)\delta_x\right]\right\}\end{gathered}}{\begin{gathered}\left[1 + \left(\frac{1}{6} - \frac{r\theta_2}{\mathrm{Re}}\right)\delta_y^2\right] \left\{1 + V^n_{i,j}\left[\frac{1}{6}\delta_x^2 (V^n_{i,j})^{-1} + \frac{rh}{2}\theta_1\delta_y\right]\right\} \\ \times \left[1 + \left(\frac{1}{6} - \frac{r\theta_4}{\mathrm{Re}}\right)\delta_x^2\right] \left\{1 + U^n_{i,j}\left[\frac{1}{6}\delta_x^2 (U^n_{i,j})^{-1} + \frac{rh}{2}\theta_3\delta_x\right]\right\}\end{gathered}}.$$

By suing the von Neumann criterion of stability [12], it is found that the diagonal elements of the amplification matrix Q have values less than unity for $\theta_i \geq 1/2,\ i = 1,2,3,4,$ and the scheme is unconditionally stable for $\theta_i \geq 1/2.$ It has an accuracy of first order with respect to time and second order with respect to space.

Numerical Results and Discussion :

In two-dimensional steady solutions of Burgers' equations, it is convenient to construct solutions of

$$\psi_{xx} + \psi_{yy} = 0\,, \tag{24}$$

that is, the steady part of the equation $\psi_t = \psi_{xx} + \psi_{yy}$.

Subsequently, by making use of the Hopf-Cole transformations, in two space dimensions [3,6], given by

$$u = -\frac{2}{\mathrm{Re}}\frac{\partial}{\partial x}(\ln\psi) = -\frac{2}{\mathrm{Re}}\frac{\psi_x}{\psi}\,,$$

$$v = -\frac{2}{\mathrm{Re}}\frac{\partial}{\partial y}(\ln\psi) = -\frac{2}{\mathrm{Re}}\frac{\psi_y}{\psi}\,, \tag{25}$$

we get exact solutions of the two-dimensional Burgers' equations. To generate an exact solutions with a shock-like structure, the following general solution of equation (24) suggests itself [7],

$$\psi = a_0 + a_1 x + a_2 y + a_3 xy + a_4\{\exp(\omega) + \exp(-\omega)\}\cos(\alpha y)\,,$$

where $a_0, a_1, a_{2,} a_3, a_4, \alpha$ and x_0 can be chosen to give specific features to the flow, and $\omega = \alpha(x - x_0)$. The application of the transformations (25) produces the following steady state solutions of the modified coupled Burgers' equations, given by (3).

$$u = -\frac{2}{\mathrm{Re}}\frac{a_1 + a_3 y + \alpha a_4\{\exp(\omega) - \exp(-\omega)\}\cos(\alpha y)}{a_0 + a_1 x + a_2 y + a_3 xy + a_4 \exp(\omega) + \exp(-\omega)\sin(\alpha y)}\,, \tag{26}$$

$$v = -\frac{2}{\mathrm{Re}}\frac{a_2 + a_3 y - \alpha a_4\{\exp(\omega) - \exp(-\omega)\}\sin(\alpha y)}{a_0 + a_1 x + a_2 y + a_3 xy + a_4 \exp(\omega) + \exp(-\omega)\cos(\alpha y)}\,. \tag{27}$$

The modified coupled Burgers' equations (3) are solved by utilizing the aforementioned multi-step finite difference scheme, developed in Section 2, with Dirichlet boundary conditions given by (26) and (27) for appropriate choices of the parameters $a_0, a_1, a_{2,} a_3, a_4, \alpha$ and x_0 and plotted.

Figures 1 and 2 show the plotting of u and v , respectively, for the following values of the parameters:

$$a_0 = a_1 = 1.1013; \quad a_2 = a_3 = 0; \quad a_4 = 1.0;$$
$$\alpha = 0.5; \quad x_0 = 1.0; \quad \mathrm{Re} = 1.0,$$

with mesh sizes $h = 1/20$ and $r = 1/2$. This is the steady state solution of equations (3), which is achieved at time $t = 0.10$ sec. It is clear from these figures that initial discontinuity is there in one corner for u, whereas for v a moderate internal gradient is present throughout the solution domain. By varying the mesh sizes and parameter $r = k/h^2$, we observed that the results remain with the limit of accuracy of the method.

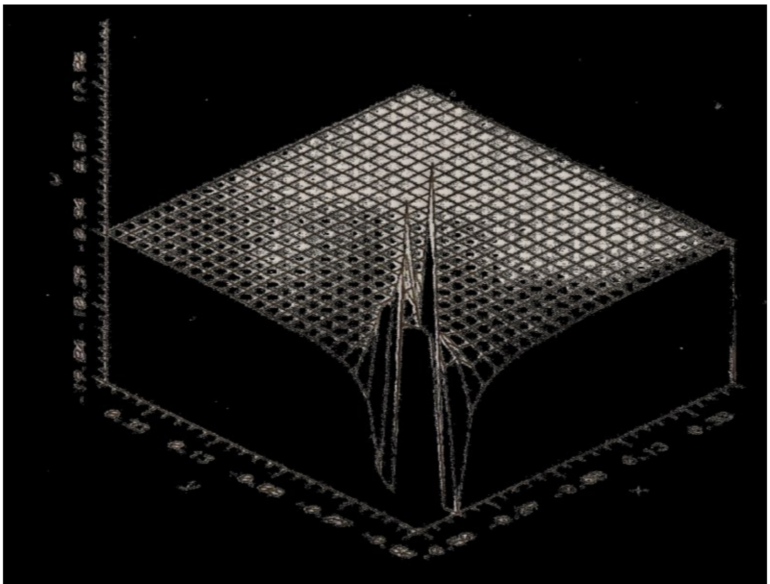

Fig. 1 Behaviour of u at Steady State for First Set of Parameter Values.

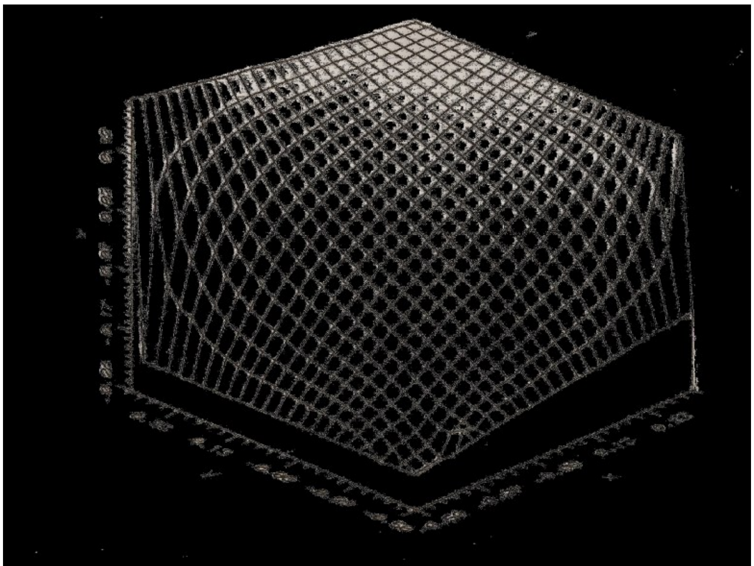

Fig. 2 Behaviour of v at Steady State for First Set of Parameter Values.

Figures 3 and 4 show the results for the values of the parameters given by:

$$a_0 = a_1 = 110.13; \quad a_2 = a_3 = 0; \quad a_4 = 1.0;$$

$$\alpha = 2.0; \quad x_0 = 1.0; \quad \text{Re} = 1.0.$$

From these figures it is clear that there is a moderate internal gradient for u, whereas for v there is a cusp type behavior at one end of the solution. Here, the steady state solution is achieved at $t = 0.15$ sec.

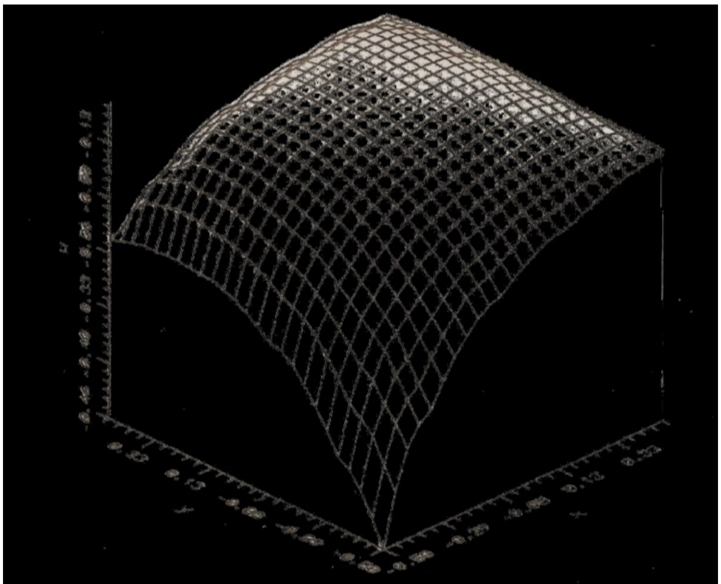

Fig. 3 Behaviour of u at Steady State for Second Set of Parameter Values.

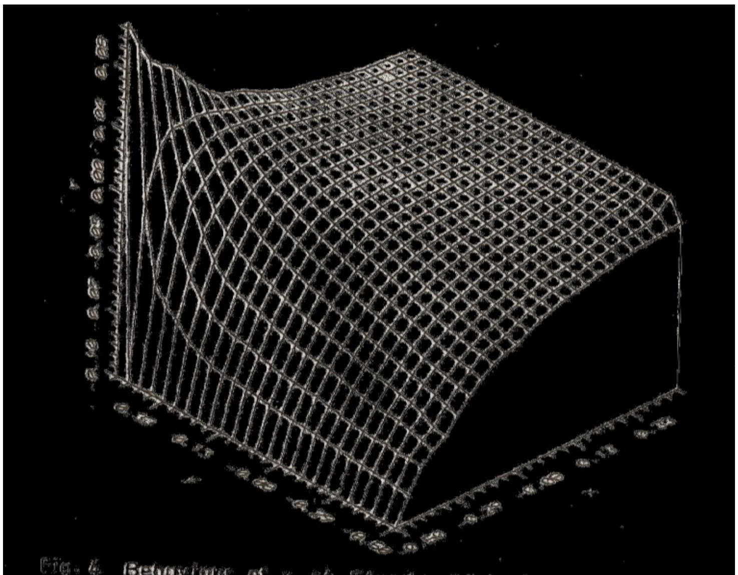

Fig. 4 Behaviour of v at Steady State for Second Set of Parameter Values.

Figures 5 and 6 show the behavior of the solution for the following values of the parameters:

$$a_0 = a_1 = 1.1013 \times 10^{13};\ a_2 = a_3 = 0;\ a_4 = 1.0;$$

$$\alpha = 18.0;\quad x_0 = 1.0;\quad \mathrm{Re} = 500\,.$$

This behavior is a moderate boundary gradient. In this case the steady state solution is achieved at $t = 0.083$ sec.

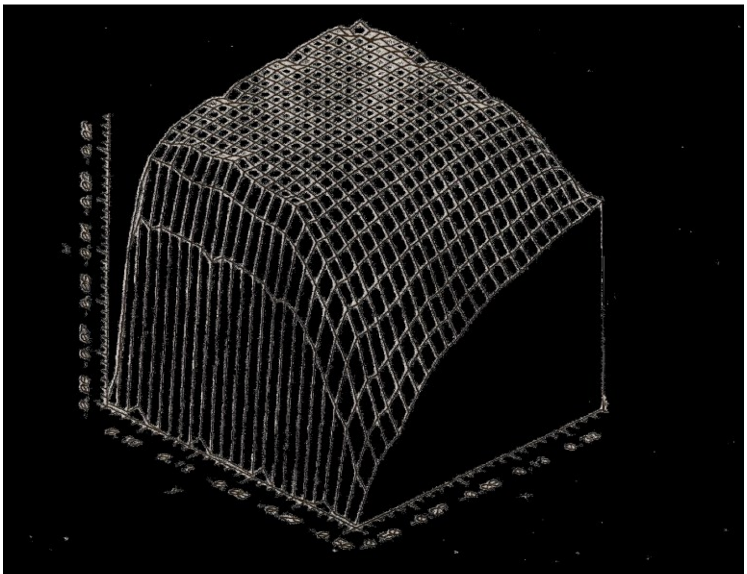

Fig. 5 Behaviour of u at Steady State for Third Set of Parameter Values.

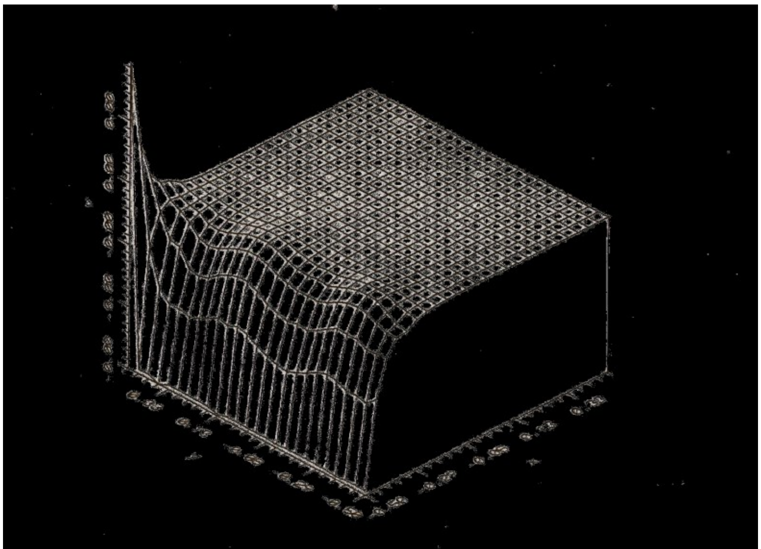

Fig. 6 Behaviour of v at Steady State for Third Set of Parameter Values.

Conclusion:

Burgers' equation was chosen since it is widely used in the literature as a model to test computational schemes intended for fluid flow problems that exhibit shocks (or severe gradients). At the present paper, the modified coupled Burgers' equations have been studied numerically by developing a multi-step finite difference scheme based on the splitting-up technique and cubic spline approximations. The stability and accuracy of the scheme are analyzed. It is found that the scheme is unconditionally stable and has an accuracy of first-order with respect to time and second-order with respect to space direction. The numerical results are obtained with severe / moderate gradients in the initial and boundary conditions. The exact steady state solutions for u and v are also plotted with the help of Maple 13 software by taking difference values of the parameters. It is concluded that the scheme is computationally efficient and produces satisfactory results even in the case of very severe gradient in the solution, and is applicable at both low and high Reynolds numbers. The proposed scheme is of a general nature which provides a wider application in the solution of non-linear problems arising in mechanics and other areas.

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طريقة فصل المؤثرات وتقريبات الأسبيلاين التكعيبي لحل معادلتي برجر المتقارنتان

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ملخص

في هذا البحث تم تطوير خطة عددية بواسطة الفروق المنتهية عن طريق فصل المؤثرات وتقريبات الاسبلاين التكعيبي، وذلك لدراسة الصيغة المعدلة لحل معادلتي برجر المتقارنتان. الخطة العددية تم تحليلها من ناحية تماسكها وتقاربها واستقراريتها. كذلك فإن فعالية تطوير هذه الخطة العددية تم مناقشتها ووجد أن نتائج هذه الخطة العددية أنها ذات رتبة اولى في الدقة عند الزمن وذات رتبة ثانية في الدقة في اتجاه الفضاء ومستقرة دون شروط. النتائج العددية أخدت بتدرج شديد ومعتدل وفقا للشروط الابتدائية والحدودية لحالة الاستقرار للحلول المرسومة باستخدام برمجية المبل 13 وذلك باخد قيم مختلفة للبارمترات المأخوذة. ونستخلص من ذلك أن الخطة العددية المقترحة هنا أعطت نتائج جيدة، حتى في حالة الحل للتدرج الشديد جدا، وأنها مناسبة لرقم رينولد في الحالتين العالية و المنخفضة. أيضاً الطبيعة العامة لهذه الخطة أعطت تطبيق واسع في حل مشكلة اللاخطية التي تظهر في الميكانيكا والمجالات الأخرى.

كلمات مفتاحية: الفروق المنتهية، فصل المؤثرات، الاسبلاين التكعيبي، معادلتي برجر المتقارنتان.